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published in
Economics Letters
1998

DOI (link to publisher)
[10.1016/S0165-1765\(98\)00081-0](https://doi.org/10.1016/S0165-1765(98)00081-0)

[Link to publication in VU Research Portal](#)

citation for published version (APA)

Houba, H. E. D., Tieman, A. F., & Brinksma, R. (1998). The Nash Bargaining Solution for Decision Weight Utility Functions. *Economics Letters*, 60, 41-48. [https://doi.org/10.1016/S0165-1765\(98\)00081-0](https://doi.org/10.1016/S0165-1765(98)00081-0)

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The Nash bargaining solution for decision weight utility functions

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Received 15 April 1997;
accepted 3 March 1998.
Available online 28 July 1998.

Abstract

A simple derivation of the non-expected utility Nash outcome, as defined in [Rubinstein et al. \(1992\)](#) [Rubinstein, A., Safra, Z., Thomson, W., 1992. On the interpretation of the Nash bargaining solution and its extension to non-expected utility preferences. *Econometrica*, 60, 1171–1186] is given for the class of decision weight utility functions. Conditions for existence and uniqueness are provided.

Author Keywords: Axiomatic bargaining; Non-expected utility; Nash solution

JEL classification codes: C78

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1. Introduction

In [Rubinstein et al. \(1992\)](#), RST hereafter, the Nash solution is redefined in terms of non-expected utility (NEU) preferences. This new definition has a very attractive interpretation, but it is stated in terms of preferences. This note aims at providing results for the NEU Nash solution stated in utility function form that will be useful in applications and make it easier to work with NEU preferences.

The class of NEU preferences considered in this note are rank dependent or decision weight (DW) utility function preferences, as defined in e.g., [Quiggin \(1982\)](#) or [Tversky and Wakker \(1995\)](#). This class is rich enough to capture most of the observed violations of the expected utility axioms.

[Grant and Kajii \(1995\)](#) characterize the Nash solution for the class of disagreement linear (DL) utility functions. The classes of DL and DW utility functions partly overlap. So, our results extend the results known for DL utility functions into the class of DW utility functions. General NEU functions, including DW utility functions, and n players, $n \geq 2$, are considered in [Grant and Kajii \(1994\)](#), GK94 hereafter. Their main results are obtained by imposing conditions such that the separating hyperplane (SH) theorem can be applied. In GK94 the main results are applied to the class of DW utility functions. In this note we make the following points. First, the Nash solution is formulated in DW utility function form. Second, we derive a simple necessary condition which generalizes the Nash solution formula for DW utility functions in GK94. Our derivation is very simple. Next, we discuss the consequences of applying the SH theorem. As an alternative, we derive a less restrictive sufficient condition in terms of first derivatives only. This means that no conditions need to be imposed on second derivatives, as is done in GK94. Finally, we provide an example that shows that the conditions imposed in GK94 are unduly restrictive.

2. The bargaining problem

Consider the bargaining problem over one dollar with NEU preferences, i.e., the set X of alternatives is equal to $\{x \in$

\mathbb{R}

$^2 | x_1 + x_2 = 1\}$, the NEU preference relation of player i , $i=1, 2$, over lotteries is \succsim_i , and the disagreement outcome is $0=(0,0)$. The NEU definition of the Nash solution is

stated in terms of lotteries with at most two prizes. Denote by $p \cdot x$ the lottery which assigns probability p to the partition $x \in X$, and the probability $1-p$ to the disagreement outcome. Each player's NEU preference relation \succsim_i , $i=1, 2$, is represented by a DW utility function. This means that the NEU function for player i , $i=1, 2$, over the lottery $p \cdot x$ has the form $w_i(p) u_i(x_i)$, where $u_i(0)=0$ is normalized and the decision weight function w_i satisfies $w_i(0)=0$ and $w_i(1)=1$. We assume that u_i , $i=1, 2$, is *strictly* increasing and continuously differentiable in x_i and that w_i is *strictly* increasing in p and continuously differentiable. Thus, $u_i'(x_i) > 0$ and $0 < w_i'(p) < \infty$. Therefore, the inverse function $w_i^{-1}(q)$ of $w_i(p)$, $i=1, 2$, exists, is strictly increasing and differentiable. The sufficient conditions imposed in GK94 are u_i , $i=1, 2$, log-concave and w_i convex and log-concave. At this point we give the NEU definition of the Nash solution as proposed in RST.

Definition 1. The partition $x^* \in X$ is a Nash outcome if for every $p \in [0, 1]$ and $x \in X$:

$$[p \cdot x \succsim_1 1 \cdot x^* \Rightarrow p \cdot x^* \succeq_2 1 \cdot x] \text{ and } [p \cdot x \succsim_2 1 \cdot x^* \Rightarrow p \cdot x^* \succeq_1 1 \cdot x].$$

3. Main results

The first result reformulates definition 1. Suppose $x^* = (x_1^*, 1-x_1^*)$ satisfies definition 1. Then we define the continuous function

(1)

$$f(x_1; x_1^*) = \begin{cases} w_2^{-1} \left(\frac{u_2(1-x_1^*)}{u_2(1-x_1)} \right) - w_1^{-1} \left(\frac{u_1(x_1)}{u_1(x_1^*)} \right), & x_1 \in [0, x_1^*), \\ 0 & x_1 = x_1^*, \\ w_1^{-1} \left(\frac{u_1(x_1^*)}{u_1(x_1)} \right) - w_2^{-1} \left(\frac{u_2(1-x_1)}{u_2(1-x_1^*)} \right), & x_1 \in (x_1^*, 1]. \end{cases}$$

The following proposition states the necessary and sufficient condition for x^* to be a Nash outcome.

Proposition 1. The partition $x^* \in X$ is a Nash outcome iff $f(x_1; x_1^*) \geq 0$ for all $x_1 \in [0, 1]$.

Proof.¹ Consider $x = (x_1, 1-x_1)$, $x_1 \in (x_1^*, 1]$. Then for all $p \in [0, 1]$, $p \cdot x \prec_2 1 \cdot x^*$ and, therefore, for all $p \in [0, 1]$ the condition $[p \cdot x \succsim_2 1 \cdot x^* \Rightarrow p \cdot x^* \succsim_1 1 \cdot x]$ holds trivially. Next, it holds that

(2)

$$p \cdot x \succsim_1 1 \cdot x^* \Leftrightarrow w_1(p)u_1(x_1) \geq u_1(x_1^*) \Leftrightarrow p \geq w_1^{-1} \left(\frac{u_1(x_1^*)}{u_1(x_1)} \right)$$

and

(3)

$$p \cdot x^* \geq_2 1 \cdot x \Leftrightarrow w_2(p)u_2(1 - x_1^*) \geq u_2(1 - x_1) \Leftrightarrow p \geq w_2^{-1} \left(\frac{u_2(1 - x_1)}{u_2(1 - x_1^*)} \right).$$

The condition $[p \cdot x \succsim_1 1 \cdot x^* \Rightarrow p \cdot x^* \geq_2 1 \cdot x]$ requires that

$$w_2^{-1} \left(\frac{u_2(1 - x_1)}{u_2(1 - x_1^*)} \right) < w_1^{-1} \left(\frac{u_1(x_1^*)}{u_1(x_1)} \right)$$

for $x_1 \in (x_1^*, 1]$. Similar arguments yield the stated expression in case $x_1 \in [0, x_1^*)$. \square

Corollary 2. If $w_1(p) = w_2(p)$, then $x^* \in X$ is a Nash outcome iff

$$u_1(x_1^*)u_2(1 - x_1^*) \geq u_1(x_1)u_2(1 - x_1).$$

Note that the corollary is valid for all expected utility functions, because $w_1(p) = w_2(p) = p$ in this case.

Since f is differentiable ($x_1 \neq x_1^*$) and $f(x_1^*; x_1^*) = 0$ it is obvious that a necessary condition for $f(x_1; x_1^*) \geq 0$ is that the left and right derivative in $x_1 = x_1^*$ are non-positive respectively non-negative. The first derivative of f is given by

(4)

$$f'(x_1; x_1^*) = \begin{cases} \frac{u_1(1 - x_1^*)u_2(1 - x_1)}{w_1' \left(w_1^{-1} \left(\frac{u_1(x_1^*)}{u_1(1 - x_1^*)} \right) \right) u_1^2(1 - x_1)} - \frac{u_1'(x_1)}{w_1' \left(w_1^{-1} \left(\frac{u_1(x_1^*)}{u_1(x_1^*)} \right) \right) u_1(x_1^*)}, & x_1 \in [0, x_1^*) \\ \frac{-u_1(x_1^*)u_2'(x_1)}{w_1' \left(w_1^{-1} \left(\frac{u_1(x_1^*)}{u_1(x_1^*)} \right) \right) u_1^2(x_1)} + \frac{u_2'(1 - x_1)}{w_2' \left(w_2^{-1} \left(\frac{u_2(1 - x_1)}{u_2(1 - x_1^*)} \right) \right) u_2(1 - x_1^*)}, & x_1 \in (x_1^*, 1] \end{cases}$$

The following proposition reformulates the necessary condition. It extends the formula for DW utility functions in GK94.

Proposition 3. If $x^* \in X$ is a Nash outcome, then x_1^* is a solution to the first order condition of the function

$$[u_1(x_1)]^{w_2'(1)} [u_2(1 - x_1)]^{w_1'(1)}$$

Proof. Combining the expressions

$$\lim_{x_1 \uparrow x_1^*} f'(x_1; x_1^*) \leq 0$$

and

$$\lim_{x_1 \rightarrow x_1^*} f'(x_1; x_1^*) \geq 0$$

yields

$$w_2'(1) \frac{w_1'(x_1^*)}{w_1(x_1^*)} - w_1'(1) \frac{w_2'(1-x_1^*)}{w_2(1-x_1^*)} = 0.$$

Hence, x_1^* is a solution to the first order condition of $w_2'(1) \ln(u_1(x_1)) + w_1'(1) \ln(u_2(1-x_1))$. \square

Note that the arguments in the proof imply that

$$\lim_{x_1 \rightarrow x_1^*} f'(x_1; x_1^*) = \lim_{x_1 \rightarrow x_1^*} f'(x_1; x_1^*) = 0$$

and, hence, f is also differentiable in $x_1 = x_1^*$ and $f'(x_1^*; x_1^*) = 0$.

If we additionally impose u_i , $i=1, 2$, is log-concave, then x_1^* is the unique interior maximizer of

$$[w_1(x_1)]^{w_2'(1)} [w_2(1-x_1)]^{w_1'(1)}$$

, which is the formula derived in GK94 for DW utility functions under additional restrictions on w_i , $i=1, 2$. It is strikingly similar to the standard asymmetric Nash solution for expected utility preferences (e.g., [Kalai, 1977](#) or [Roth, 1979](#)). The reason that the derivatives of w_1 and w_2 in $p=1$ uniquely determine the weights is as follows. In RST it is argued that the Nash solution should be robust in the sense that every objection made by one of the players should be credibly counter objected by the other player. In particular, this should hold if the objecting player wants an incremental higher share. For such an incremental extra share this player can only afford a low risk of breakdown, i.e., $p \approx 1$. Similarly, for the other player, who can only afford credible counter objections with a low risk of breakdown.

Since $w_i'(1) > 0$, $i=1, 2$, every candidate Nash outcome is interior, i.e., $x_1^* \in (0, 1)$. This implies that each player can successfully counter object against objections in which his opponent claims (almost) the entire cake, because $f(0; x_1^*)$, $f(1; x_1^*) > 0$. Thus, only modest objections matter.

In GK94 the SH theorem is applied in order to provide sufficient conditions for existence of a Nash outcome. [Fig. 1](#) illustrates the application of the SH theorem in the (x_1, p) -space for

$$w_1(p)w_2(x_2) = (p/1 + (1-p)) x_2$$

and for

$$x_1 > x_1^* = \frac{1}{2}$$

. The condition $f(x_1; x_1^*) \geq 0$ requires that

$$w_1^{-1} \left(\frac{w_1(x_1^*)}{w_1(x_1)} \right)$$

lies above

$$w_2^{-1} \left(\frac{w_2(1-x_1)}{w_2(1-x_1^*)} \right).$$

Application of the SH theorem imposes that

$$w_1^{-1} \left(\frac{w_1(x_1^*)}{w_1(x_1)} \right)$$

is convex and

$$w_2^{-1} \left(\frac{w_2(1-x_1)}{w_2(1-x_1^*)} \right)$$

is concave. The separating hyperplane is the line given by

$$p = -\frac{w_1'(x_1^*)}{w_1'(1)w_1(x_1^*)} (x_1 - x_1^*) + 1$$

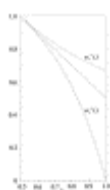
through the point $(x_1, p) = (x_1^*, 1)$. Clearly, this line separates the two curves, which suffices to obtain $f(x_1; x_1^*) \geq 0$. However, [Fig. 2](#), which is based on

$$w_1(p)w_2(x_2) = \frac{p}{1 - \frac{1}{2}(1-p)} x_2$$

illustrates a case in which $f(x_1; x_1^*) \geq 0$ while

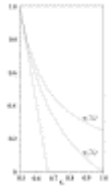
$$w_2^{-1} \left(\frac{w_2(1-x_1)}{w_2(1-x_1^*)} \right)$$

convex.



[Full-size image](#) (4K)

Fig. 1. For $\beta_1=\beta_2=1$ in Example 1 the SH theorem applies.



[Full-size image](#) (4K)

Fig. 2. For $\beta_1=\beta_2=-$



in Example 1 the SH theorem does not apply.

Application of the SH theorem implies f is convex. A less restrictive sufficient condition for $f(x_1; x_1^*) \geq 0$ requires that $f(x_1; x_1^*)$ is non-increasing on $[0, x_1^*)$ and non-decreasing on $(x_1^*, 1]$, which includes the case f convex. In order to do so we define the boldness $b_1(x_1)$ of player 1 as

$$b_1(x_1) = \begin{cases} \frac{w_1^{-1}\left(\frac{w_1(x_1)}{w_1(x_1^*)}\right)}{w_1^{-1}\left(\frac{w_1(x_1)}{w_1(x_1^*)}\right)} w_1(x_1), & x_1 \in [0, x_1^*], \\ \frac{w_1^{-1}\left(\frac{w_1(x_1)}{w_1(x_1^*)}\right)}{w_1^{-1}\left(\frac{w_1(x_1)}{w_1(x_1^*)}\right)} w_1(x_1), & x_1 \in (x_1^*, 1], \end{cases}$$

and the boldness $b_2(x_1)$ of player 2 as

$$b_2(x_1) = \begin{cases} \frac{w_2^{-1}\left(\frac{w_2(1-x_1)}{w_2(1-x_1^*)}\right)}{w_2^{-1}\left(\frac{w_2(1-x_1)}{w_2(1-x_1^*)}\right)} w_2(1-x_1), & x_1 \in [0, x_1^*], \\ \frac{w_2^{-1}\left(\frac{w_2(1-x_1)}{w_2(1-x_1^*)}\right)}{w_2^{-1}\left(\frac{w_2(1-x_1)}{w_2(1-x_1^*)}\right)} w_2(1-x_1), & x_1 \in (x_1^*, 1]. \end{cases}$$

The boldness $b_i(x_1)$, $i=1, 2$, can be regarded as a measure of the willingness to accept the risk $w_i^{-1}(\cdot)$ of disagreement in return for an improvement of the outcome of the magnitude $|x_1 - x_1^*|$. This definition extends the definition of marginal boldness in GK94, where the latter is defined as $b_i(x_1^*)$, $i=1, 2$.

Proposition 4. A sufficient condition for $x^* \in X$ to be a Nash outcome is

$$x_1 \in [0, x_1^*] : f'(x_1; x_1^*) \leq 0 \Leftrightarrow \frac{u_2(1-x_1^*)}{u_2(1-x_1)} \cdot b_2(x_1) \leq \frac{u_1(x_1)}{u_1(x_1^*)} \cdot b_1(x_1),$$

$$x_1 \in (x_1^*, 1] : f'(x_1; x_1^*) \geq 0 \Leftrightarrow \frac{u_2(1-x_1)}{u_2(1-x_1^*)} \cdot b_2(x_1) \geq \frac{u_1(x_1^*)}{u_1(x_1)} \cdot b_1(x_1).$$

The interpretation of these conditions is as follows. Suppose $x_1 < x_1^*$. Then

$$\frac{u_2(1-x_1^*)}{u_2(1-x_1)} \cdot b_2(x_1)$$

relates player 2's boldness to the size of the intended gain of the objection x_1 over x_1^* . The sufficient condition states (in terms of first derivatives) that player 2 should not object if player 1 is willing to counterobject. The sufficient condition only imposes conditions on first derivatives and not on second derivatives.

The following example illustrates that the sufficient conditions in GK94 are more restrictive than the conditions in proposition 4.

Example 1. Consider the disappointment averse utility function

$$u_i(p)u_i(x_i) = \frac{p}{1 + (1-p)\beta_i} x_i, \beta_i > -1$$

and $i=1,2$, as axiomatized in [Gul \(1991\)](#). We will show that

$$x_1^* = \frac{1 + \beta_2}{2 + \beta_1 + \beta_2}$$

is the unique Nash outcome for every pair (β_1, β_2) . First, application of Proposition 3 yields that

$$x_1^* = \frac{1 + \beta_2}{2 + \beta_1 + \beta_2}$$

is the unique maximizer of

$$x_1^{1+\beta_2} x_2^{1+\beta_1}$$

. Second, we have to verify whether $f(x_1; x_1^*) \geq 0$. Suppose $x_1 > x_1^*$. Then

$$\frac{f(x_1; x_1^*)}{u_1(x_1) + \beta_1 u_1(x_1^*)} - \frac{(1 + \beta_2)u_2(1 - x_1)}{u_2(1 - x_1^*) + \beta_2 u_2(1 - x_1)} \geq 0$$

can be rewritten as

$$(1 + \beta_1) \left(x_1 - \frac{1 + \beta_2}{2 + \beta_1 + \beta_2} \right)^2 \geq 0, \quad x_1 \in (x_1^*, 1].$$

A similar quadratic expression follows for $x_1 < x_1^*$. Hence, x_1^* is the Nash outcome for all $\beta_1, \beta_2 > -1$.

The sufficient conditions in GK94 imply $w_i(p)$, $i=1, 2$, is convex and log-concave, i.e., $\beta_i \in [0, 1]$. Thus, for $\beta_1, \beta_2 \in (-1, 0)$, this example shows that a unique Nash outcome exists for concave w_i . For $\beta_1, \beta_2 \in (1, \infty)$ this example shows a unique Nash outcome exists for convex w_i that are not log-concave.

4. Concluding remarks

Our results imply the following two-step procedure to calculate Nash outcomes for specific functional forms. First, compute all candidates for the Nash outcome by finding the stationary points of Proposition 3. Second, verify for each candidate Nash outcome whether the condition in either Proposition 1 or Proposition 4 is met. This latter step is equivalent to evaluating whether a particular function is non-negative. In case of a numerical representation one can resort to computer assistance in verifying these conditions.

The sufficient conditions in GK94 are conditions imposed upon each player's DW utility function separately. The condition in Proposition 4 restricts the function f and, therefore, this is a condition upon both player's DW utility functions simultaneously. Therefore, the latter approach is less demanding. Nevertheless, the condition in Proposition 4 is still restrictive, because cases in which the function f has local minima x_1 other than x_1^* as well as cases with interior local maxima $x_1 \in (0, 1)$ are excluded.

Acknowledgements

The authors thank Gerard van der Laan, Gijs van Lomwel, Hans Peters and Peter Wakker for useful comments.

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¹This proof only requires u_i , $i=1, 2$, and w_i strictly increasing.